УДК 519.1:519.2:519.6:519.8(075).

TWO DIMENSIONAL BINARY STATES MOORE ANDVON NEUMANN CELLULAR AUTOMATA WITH NULLBOUNDARY

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It is known that cellular automata (CA) theory is a very rich and useful dynamical model by focusing on their local information and neighboring cells. The fundamental structure of CA is a discrete special dynamical model, but the global behaviors at many iterative times can be close nearly a continuous mathematical model and system. The mathematical view of the basic model shows the computable values of the mathematical structure of CA. In the present paper, it is investigated the structure of two-dimensional (2D) finite, linear, Moore and von Neumann CA with null boundary over Galois field *GF*(2). In other words, it is considered on Galois field, i.e. 2-state (binary) case or Z_2. Here we obtain the transition or information rule matrices for each special Moore and von Neumann linear cases presented in the paper. The determination of the structure problem of special type of cellular automaton is studied by means of the matrix algebra theory. These types of special linear 2D cellular automata can find many different real life applications in special case situations, e.g. image processing area, textile design, video processing, DNA research, etc.

***Keywords*:** two-dimensional (2D) finite linear cellular automata, null boundary, Galois field, fundamental structure of cellular automata, Moore and von Neumann linear cases, matrix algebra theory.

Теория клеточных автоматов (КА) представляет собой особую динамическую модель, фокусирующуюся на локальной информации с соседними клетками. Структура КА способна двигаться вперед и назад по КА, чтобы распознать их поведение. Хотя КА является дискретной динамической моделью, глобальное поведение во многих итеративных ситуациях может быть близко к непрерывной математической системе. Математическая модель КА показывает вычислимые значения его динамической структуры. В данной работе исследован теоретический подход к двухмерной (2D) гибридной линейной КА с периодическими граничными условиями в случае трех состояний, т.е. $\mathbf{Z}_3$ или трехмерного поля. Была построена матрица правил перехода для 2D гибридного линейного КА с этими особыми граничными условиями с помощью теории матричной алгебры. В ближайшем будущем, эти типы специальных КА и их математические представления могут быть найдены во многих различных реальных приложениях в особых ситуациях, например, в теории вычислимости, теоретической химии и биологии, областях обработки

изображений, текстильном дизайне. В этой статье мы сконцентрировали специальное семейство (правило 9840 и правило 9841) 2D конечных линейно гибридных клеточных автоматов с периодическим состоянием на поле $\mathbf{Z}_3$. Здесь мы изучаем специфическую связь между гибридными клеточными автоматами и характеристикой 2D гибридного КА с периодическими граничными условиями. Исследуется определение задачи характеризации этого специального клеточного автомата с помощью теории матричной алгебры. Благодаря КА очень просто объяснить некоторые важные математические исследования, а также объяснить очень сложные состояния хаоса в динамических системах. Важно отметить, что мы работаем с КА, созданным гибридным правилом над полем $\mathbf{Z}_3$ и находим матрицы правил T_{Rules}, соответствующие конечной 2D линейно-гибридной КА, после чего представляем характеристику этих правил как теоремы.

Ключевые слова: динамические системы, соседние клетки, автоматы, поведение, математическая модель, итеративное время, структура КА, теория матричной алгебры, правило 9840, правило 9841, линейно-гибридные клеточные автоматы.

Kletkali avtomatlar (KA) nazariyasi qo'shni kletkalar bilan lokal ma'lumotlarga fokuslanuvchi maxsus dinamik modeldir. Kletkali avtomatlarning xatti-harakatlarini aniqlash, ularning kletkalar bo`ylab oldinga va orqaga siljishlari orqali amalga oshiriladi. KA diskret dinamik model bo'lsa ham, ko'p iterative vaziyatlarda global xatti-harakatlari uzluksiz matematik tizimga yaqin bo'lishi mumkin. KA ning matematik modeli uning dinamik tuzilmasini hisoblaydigan qiymatlarini ko'rsatadi. Ushbu maqolada $\mathbf{Z}_3$ yoki uch o'lchovli maydonda davriy chegara shartlariga ega bo'lgan ikki o'lchovli (2D) chiziqli gibrid KA ning nazariy yondashuvi o'rganilgan. Matritsalar algebrasi nazariyasi yordamida, maxsus chegaraviy shartlarga ega ikki o'lchovli (2D) chiziqli gibrid KA uchun o`tish qoidalari matritsasi qurilgan. Yaqin kelajakda ushbu turdagi maxsus KA va ularning matematik tadbiqlari alohida vaziyatlarda, masalan, hisoblash nazariyasi, nazariy kimyo va biologiya, tasvirlarni qayta ishlash sohalari, to'qimachilik dizayni va boshqalarda toppish mumkin. Ushbu maqolada biz $\mathbf{Z}_3$ maydonda davriy holatga ega ikki o'lchivli (2D) chekli chiziqli gibrid KA maxsus oilasiga (9840 qoida va 9841 qoida) to'xtalib o`tdik. Bunda biz gibrid kletkali avtomatlar va davriy chegaraviy shartga ega 2D gibrid xarakteristikali KA o'rtasidagi o'ziga xos bog'liqlikni o'rganamiz. Ushbu maxsus KA ni matritsalar algebrasi nazariyasidan foydalangan holda xarakterlash masalasini aniqlandi.

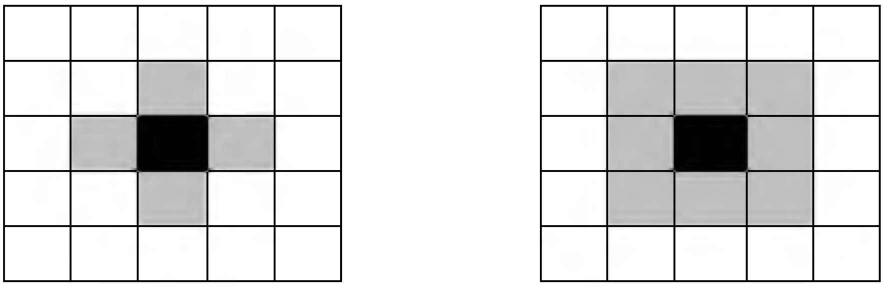
***Tayanch iboralar*:** dinamik tizimlar, qo'shni hujayralar, avtomatika, matematik model, iterative vaqt, CA tuzilishi, matritsa algebrasi nazariyasi, 9840 qoida, 9841 qoida, chiziqli gibrid CA.

I. INTRODUCTION

Cellular automata theory (CA theory for brevity) introduced by Ulam and von Neumann [1]in the early 1950s and was systematically studied by Hedlund from mathematical perspective. One-dimensional (1D) CA has been investigated to very point of views. On the other hand a little interest was given to two-dimensional cellular automata (2D CA). Von Neumann [1] showed that a cellular automaton could have universal properties. Due to complexity of CA theory, von Neumann rules were never studied on a computer language. In the beginning of 1980s, Wolfram [2] studied in very details of a family of simple 1D CA rules and showed that even these simplest rules are capable of interesting complex behaviors. Some basic and original mathematical CA models using matrix algebra over the two states or binary field which characterize the behavior of 2D nearest neighborhood linear CA with null and periodic boundary conditions have been seen in the literature [7, 8, 9, 10, 11, 12]. 2D CA theory has received remarkable interest and attention in the last few decades [3, 4, 5, 6, 7, 12, 13, 14,15, 16, 17]. Due to its striking structures, CA theory has given the opportunity to model and understand many interesting behaviors in nature easier. Here we study the theory of two dimensional uniform null boundary Moore and von Neumann CA (2D Null CA) over Galois field $GF(2)$ (see CA structures in Figs. 1-3).

In this paper, we concentrate a special family of 2D finite linear Moore and von Neumann CA with null boundary condition over Galois field, i. e, the binary states field Z_2 .We set up a specific relation between the structure of Moore and von Neumann CA and transition matrix rules of 2D linear CA with null boundary condition. We determine and study of the transition rule matrices of this Moore and von Neumann CA by means of the matrix algebra theory. It is known that CA nature is very simple to allow mathematical studies in dynamical systems, itis believed that these linear CA can be found many different kind of real life applications. The algebraic consequences of these 2D linear CA and relates some elegant real life applications can be found in the literature (see details in [12, 13, 16, 17, 18]).

The organization of the present paper is constructed as follows. In Section 2, it is given the preliminaries of CA theory and their lattice structures. Section 3 presents the transition rule matrices for each cases, corresponding to the 2D Moore and von Neumann finite CA. Finally conclusions are summarized in Section 4.



(a) (b)

Figure 1: Von Neumann and Moore neighbor hoods of 2D finite linear CA respectively.

II. MAIN PART

Preliminaries.

We study on integer lattice model, i.e. Z^2 (see Figs. 1-2), for 2-states finite linear Moore and von Neumann 2D CA. We firstly focus on special null boundary condition of 2D Moore and von Neumann CA over binary field. After then we obtain the information rule matrices for special cases. Considering the neighbors of the extreme cells, there are two well-known studied neighbor approaches as below.

1. Null or 0-fixed boundary CA (Null): The borderline cells are connected to the 0-state.
2. Periodic boundary CA (Periodic): The borderline cells are contiguous to each other periodically in the boundaries.

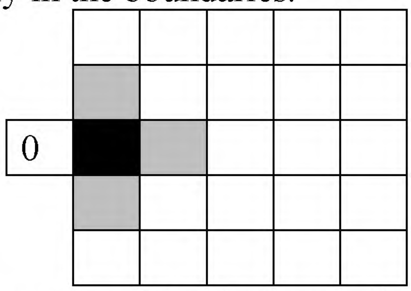


Figure 2: Null or 0-fixed boundary CA. The borderline cells are connected to the 0-state.

In the present paper we only deal with null or fixed (0-th state) valued boundary condition. In other words, it is said that CA with null boundary case is the borderline cells in the boundaries are considered as zero-fixed states. (i.e. The surrounding neighbor cells spin values are all 0-state, check for better understanding in [8, 16]).

	$2^6=64$	$2^7=128$	$2^8=256$	
	$2^5=32$	$2^0=1$	$2^1=2$	
	$2^4=16$	$2^3=8$	$2^2=4$	

Figure 3: The rule convention of two states linear 2D CA.

It is also known that the description question of 2D CA configurations can be transformed to the description of 1D configurations by considering $m \times n$ configurations as $mn \times 1$ type configurations as follows. To obtain this procedure, it is defined the following map

$$I : M_{m\times n}(Z_2) \to Z_2^{mn} \qquad (1)$$

that gets the t^{th} state $[X_t]$ given by

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \dots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix} \to \left(x_{11}, x_{12}, \dots, x_{1n}, \dots, x_{m1}, \dots, x_{mn}\right)^t . \qquad (2)$$

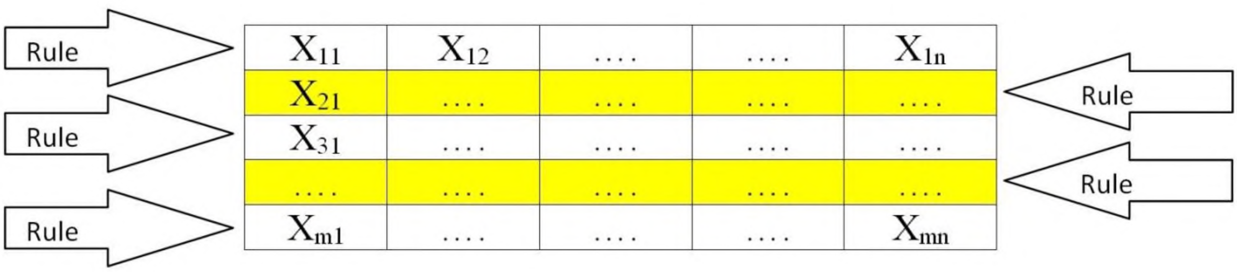


Figure 4: Rule application convention of uniform 2D CA for each rows and columns.

Note that the superscript t indicates the transpose sign. Then the local rules are assumed to act on Z_2^{mn} on the contrary $M_{m\times n}(Z_2)$. Hence the $C^{(t)}$ matrix

$$C^{(t)} = \begin{pmatrix} x_{11}^{(t)} & \dots & x_{1n}^{(t)} \\ \vdots & \dots & \vdots \\ x_{m1}^{(t)} & \dots & x_{mn}^{(t)} \end{pmatrix} \qquad (3)$$

is denoted the configuration matrix at time t for 2D finite CA. Using the equation (2), it can be defined as below

$$\left(T_{Rule}\right)_{mn\times mn}\cdot\begin{pmatrix}x_{11}^{(t)}\\ \vdots\\ x_{1n}^{(t)}\\ \vdots\\ x_{m1}^{(t)}\\ \vdots\\ x_{mn}^{(t)}\end{pmatrix}=\begin{pmatrix}x_{11}^{(t+1)}\\ \vdots\\ x_{1n}^{(t+1)}\\ \vdots\\ x_{m1}^{(t+1)}\\ \vdots\\ x_{mn}^{t+1}\end{pmatrix} \tag{4}$$

For analysis and further computation, the each cell states has a spin value which takes in finite or infinite states set. Here this spin values set is chosen from Galois field $GF(2)$.Also it is denoted by $x_{(i,j)}^{(t)}$ as the spin states of the cell in (i, j) at time t. Moreover the spin states of the cell (i, j) at time $t+1$ should be denoted by $x_{(i,j)}^{(t+1)}=y_{(i,j)}^{(t)}$. Consider the triangular transition configuration or information matrix

$$C^{(t)}=\begin{pmatrix}x_{11}^{(t)} & \dots & x_{1n}^{(t)}\\ \vdots & \dots & \vdots\\ x_{m1}^{(t)} & \dots & x_{mn}^{(t)}\end{pmatrix}.$$

If we combine planar triangular structure with column vectors by converting them from $C^{(t)}$ to $\left(\left[X\right]_{mn\times1}\right)^T=\left(x_{11}^{(t)},x_{12}^{(t)},\dots,x_{1n}^{(t)},\dots,x_{m1}^{(t)},\dots,x_{mn}^{(t)}\right)^T$, then it can be considered the transition rule matrix T_{Rule} as follows.

$$\left(T_{Rule}\right)_{mn\times mn}\cdot\left[X\right]_{mn\times1}=\left[Y\right]_{mn\times1}, \tag{5}$$

where $\left(\left[Y\right]_{mn\times1}\right)^T=\left(y_{11}^{(t)},y_{12}^{(t)},\dots,y_{1n}^{(t)},\dots,y_{m1}^{(t)},\dots,y_{mn}^{(t)}\right)^T$.

Remark 1. In the literature, the most commonly used lattice for CA is an orthogonal Z^d structure, several studies have been done to test the properties of other lattice models such as hexagonal (see [12, 13, 16, 19]). In the present work we investigate CA studies on Z^2 lattice model, for 2-state, finite, linear, Moore and von Neumann neighborhood. Also note that one can study von Neumann and Moore neighborhood any different lattice model as a future direction.

Uniform cellular automata transition rule matrices under null boundary.

In the present paper, it is dealt with special finite CA for von Neumann neighbors on triangular lattice. These CA are studied under null (or fixed 0-th state) boundary condition (Null) with the 2-state spin values, i.e. over Galois field or Z_2. In the present section, we investigate the transition rule matrix of finite null boundary CA.

Theorem 1. For uniform type of CA, the representation of the fundamental linear CA rule matrices (*Rule*2^0= *Rule* 1, *Rule* 2^1 = *Rule* 2, *Rule* 2^2 = *Rule*4, *Rule* 2^3 = *Rule* 8, *Rule* 2^4 = *Rule* 16, *Rule* 2^5 = *Rule* 32, *Rule* 2^6 = *Rule* 64, *Rule* 2^7 = *Rule* 128 and *Rule* 2^8 = *Rule* 256)with null boundary neighborhood over Z_2 is obtained considering the fixed A^N and B^N matrices presented below and is given as follows

$$Rule1Null:[X^{t+1}]=[X^t],$$
$$Rule2Null:[X^{t+1}]=[X^t][B^N],$$
$$Rule4Null:[X^{t+1}]=[A^N][X^t][B^N],$$
$$Rule8Null:[X^{t+1}]=[A^N][X^t],$$
$$Rule16Null:[X^{t+1}]=[A^N][X^t][A^N],$$
$$Rule32Null:[X^{t+1}]=[X^t][A^N],$$
$$Rule64Null:[X^{t+1}]=[B^N][X^t][A^N],$$
$$Rule128Null:[X^{t+1}]=[B^N][X^t],$$
$$Rule256Null:[X^{t+1}]=[B^N][X^t][B^N],$$

Where A^N, B^N are given as

$$A^N=\begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & ... & 0 & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix}_{n\times n} \qquad B^N=\begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix}_{n\times n}$$

Proof 1. Let us consider e_{ij} as the matrix of size $m\times n$. Here e_{ij} is the matrix whose (i, j) position element is equal to one and the other all entries elements equal to zero. This is well-known that the vectors e_{ij} present the standard basis elements for the matrix space. To establish the transition rule matrix (T_{Rule} structure, it is needed to specify the action of (T_{Rule} on the bases e_{ij} vectors elements respectively. Then we obtain the transition rules of the representation matrix presented as in theorem.

Corollary 1. The following matrix relations between the transitions rules are obtained as below.

$$(Rule2Null)^T=Rule32Null,\ (Rule4Null)^T=Rule64Null,$$
$$(Rule8Null)^T=Rule128Null,\ (Rule16Null)^T=Rule256Null,$$

where T is the matrix transpose operator.

Proof 2. One can easily seen that $(A^N)^T=B^N$. These relations can be found using this equality as given in the corollary.

Theorem 2. (General Rule Theorem for Null Case) The general transition rule matrix of 2Dlinear CA considering all primary rules (*Rule* 1, *Rule* 2, *Rule* 4, *Rule* 8, *Rule* 16, *Rule*32, *Rule* 64, *Rule* 128and *Rule* 256) under the null boundary condition can be given in the following way:

$$T_{Rule-Null}=\begin{pmatrix} P & R & 0 & ... & 0 & 0 & 0 \\ S & P & R & ... & 0 & 0 & 0 \\ 0 & S & P & R & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & S & P & R \\ 0 & 0 & 0 & 0 & ... & S & P \end{pmatrix}_{mn\times mn}$$

where P, R and S are one of the following matrices of the order $n\times n$:[0],[I],[A^N],[B^N],[$I+A^N$];[$I+B^N$];[A^N+B^N] and [$I+A^N+AB^N$].

Proof 3. Let us firstly find all the transition rule matrices of 2D linear CA (all primary rules *Rule* 1, *Rule* 2, *Rule* 4, *Rule* 8, *Rule* 16, *Rule* 32, *Rule* 64, *Rule* 128and *Rule* 256)under the null boundary condition. It can be given in the following way.

$$T_{Rule1Null}=\begin{pmatrix} I & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & I & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & I & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & ... & I & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & I \end{pmatrix}$$

$$T_{Rule2Null}=\begin{pmatrix} A^N & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & A^N & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & A^N & ... & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & 0 & A^N & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & A^N \end{pmatrix}$$

$$T_{Rule4Null}=\begin{pmatrix} 0 & A^N & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & A^N & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & A^N & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & 0 & 0 & A^N \\ 0 & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix} \quad T_{Rule8Null}=\begin{pmatrix} 0 & I & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & I & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & I & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & ... & 0 & I \\ 0 & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix}$$

$$T_{Rule16Null}=\begin{pmatrix} 0 & B^N & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & B^N & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & B^N & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & B^N \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$T_{Rule32Null}=\begin{pmatrix} B^N & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & B^N & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & B^N & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & B^N & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & B^N \end{pmatrix}$$

$$T_{Rule64Null}=\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ B^N & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & B^N & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & B^N & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & B^N & 0 \end{pmatrix} \quad T_{Rule128Null}=\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ I & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & I & 0 \end{pmatrix}$$

$$T_{Rule256Null}=\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ A^N & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & A^N & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A^N & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & A^N & 0 \end{pmatrix}$$

It is known that any linear 2D CA rules can be obtained by using the linearity of these primary rules. Hence it is found the general rule theorem of null case.

Uniform cellular automata transition rule matrices under periodic boundary.

Theorem 3. For uniform type of CA, the representation of the fundamental linear CA rule matrices (*Rule* 2^0 = *Rule* 1, *Rule* 2^1 = *Rule* 2, *Rule* 2^2 = *Rule* 4, *Rule* 2^3 = *Rule* 8, *Rule* 2^4 = *Rule* 16, *Rule* 2^5 = *Rule* 32, *Rule* 2^6 = *Rule* 64, *Rule* 2^7 = *Rule* 128 and *Rule* 2^8 = *Rule* 256) with periodic boundary neighborhood over Z_2 is obtained considering the fixed A^P and B^P matrices presented below and is given as follows

$$Rule1Periodic:[X^{t+1}]=[X^{t}],$$
$$Rule2Periodic:[X^{t+1}]=[X^{t}][B^{P}],$$
$$Rule4Periodic:[X^{t+1}]=[A^{P}][X^{t}][B^{P}],$$
$$Rule8Periodic:[X^{t+1}]=[A^{P}][X^{t}],$$
$$Rule16Periodic:[X^{t+1}]=[A^{P}][X^{t}][A^{P}],$$
$$Rule32Periodic:[X^{t+1}]=[X^{t}][A^{P}],$$
$$Rule64Periodic:[X^{t+1}]=[B^{P}][X^{t}][A^{P}],$$
$$Rule128Periodic:[X^{t+1}]=[B^{P}][X^{t}],$$
$$Rule256Periodic:[X^{t+1}]=[B^{P}][X^{t}][B^{P}],$$

Where A^P, B^P are given as

$$A^{P}=\begin{pmatrix}0&1&0&\ldots&0&0&0\\0&0&1&\ldots&0&0&0\\0&0&0&\ldots&\ldots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\ldots&\ldots&0&1\\1&0&0&0&\ldots&0&0\end{pmatrix}_{n\times n}\qquad B^{P}=\begin{pmatrix}0&0&0&\ldots&0&0&1\\1&0&0&\ldots&0&0&0\\0&1&0&\ldots&\ldots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\ldots&1&0&0\\0&0&0&0&\ldots&1&0\end{pmatrix}_{n\times n}$$

Proof 4. Let us consider e_{ij} as the standard basis elements for the matrix space. To construct the transition rule matrix T_{Rule} structure, we need to specify the action of T_{Rule} on the bases e_{ij} vectors elements respectively. Then it is obtained the transition rules of the representation matrix as presented in the theorem.

In the next subsection, it will be obtained the transition rule matrices corresponding to the2D finite Moore and von Neumann neighborhood CA local rule under null boundary condition. It has been studied two special rule cases as presented theorems below respectively.

Structure of linear Moore and Von Neumann CA rule matrices (Rule 512Null and Rule 170Null).

Considering the general case of the CA reversibility problem, the verification of reversibility of all 1D, 2D uniform or hybrid CA is a challenging problem. In other words, up to now there does not exist an algorithm for any general CA rule matrix situation. Here the significance of the reversibility phenomena of 2D linear CA, or understanding of a general case, is shortly emphasized. The kernel dimension of the rule matrix of uniform or hybrid CA presents a hint or path to check the states of the iterative phase diagram and to detect CA reversibility or non-reversibility. Hence, to obtain the kernel dimension of a 2D uniform or hybrid CA, the rule matrix rank values can be studied, i.e. the rank of $(T_{rule})_{mn\times mn}$. Considering a finite situation of uniform or hybrid CA, many researchers have studied the corresponding CA rule matrices in order to find the invertibility of a 2D finite linear-hybrid or uniform CA. Firstly the rule matrices T_{rule}, corresponding to finite 2D linear-hybrid CA, are found, then the yare characterized considering the reversibility problem of hybrid CA. The following iteration property among the column vectors X(t) can be stated, with the rule matrix T_{rule}:

$$X^{(t+1)} = T_{rule} \cdot X^{(t)} (\text{mod}2).$$

If the transition rule matrix T_{rule} is a non-singular matrix, then we have

$$X^{(t)} = (T_{rule})^{1} \cdot X^{(t+1)} (\text{mod}2).$$

Here, the main problem related to the T_{rule} is whether the rule matrix T_{rule} is an invertible matrix or not for any cases. Some know that any 2D finite CA is called a reversible CA if and only if CA rule matrices T_{rule} are non-singular (see references for extra details). If the transition matrices T_{rule} of a CA have full rank properties, then it is said to be an invertible CA. Hence the 2D finite hybrid or uniform CA is a reversible one, otherwise it is called an irreversible CA. In the present study, we work with special 2D CA defined by Moore and von Neumann linear rules over the field Z_2 under null boundary case. We will determine the transition matrix T_{rule} for Moore and von Neumann CA cases.

Theorem 4. (*Moore CA Rule Matrix = Rule* 512*Null*) For uniform type of CA, the representation of the Moore CA rule matrix (*Rule* 512*Null* = *Rule* 1+*Rule* 2+*Rule* 4+*Rule* 8+*Rule* 16 + *Rule* 32 + *Rule* 64 + *Rule* 128 + *Rule* 256) with null boundary neighborhood over Z_2 is obtained as follows

$$(T_{512Null})_{mn\times mn} = \begin{pmatrix} K & M & 0 & 0 & \dots & 0 & 0 \\ M & K & M & 0 & \dots & 0 & 0 \\ 0 & M & K & M & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M & K & M \\ 0 & 0 & 0 & 0 & \dots & M & K \end{pmatrix},$$

where partitioned matrix are given as follows.

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}_{n\times n} \text{ and} M = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix}_{n\times n}.$$

Theorem 5. (*von Neumann CA Rule Matrix = Rule* 170*Null*) For uniform type of CA, the representation of the von Neumann CA rule matrix (*Rule* 170*Null* = *Rule* 2 + *Rule* 8 +*Rule* 32+*Rule* 128) with null boundary neighborhood over Z_2 is obtained as follows

$$(T_{170Null})_{mn\times mn}=\begin{pmatrix} K & I & 0 & 0 & ... & 0 & 0 \\ I & K & I & 0 & ... & 0 & 0 \\ 0 & I & K & I & ... & 0 & 0 \\ : & : & : & : & : & : & : \\ 0 & 0 & 0 & ... & I & K & I \\ 0 & 0 & 0 & 0 & ... & I & K \end{pmatrix}$$

where partitioned matrix are given as follows.

$$K=\begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & 1 & ... & 0 & 0 \\ : & : & : & : & : & : & 1 \\ 0 & 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix}_{n\times n} \quad \text{and} \; I=\begin{pmatrix} 1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & 0 & ... & 0 & 0 \\ : & : & : & : & : & : & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 \end{pmatrix}_{n\times n}$$

is the identity matrix.

III. CONCLUSION

We investigate the mathematical basic theory of two dimensional, uniform null boundary, Moore and von Neumann CA over Galois field $GF(2)$. Due to main CA structures are sufficiently simple to investigate in mathematical ways, the present construction could be applied many areas related to these CA using any other transition rules. We present two main rule theorems (Moore and von Neumann rule) for determining the structure of Moore (Rule 512) and von Neumann (Rule 170) CA for a general case of linear transformation. Also after constructed the transition rule matrix representation of 2D linear von Neumann CA, it can befo und some real life applications for the 2D linear CA. We see that 2D CA theory could be applied successfully in especially image processing area [7, 13, 14, 15, 16, 17] and the other science branches in near future [8, 9, 11]. Some other interesting results and further connections on this direction wait to be explored in Moore and von Neumann 2D CA [7, 19].

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